A catalogue of designs for partial profiles in paired comparison experiments with three groups of factors

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A common strategy for avoiding information overload in multi-factor paired comparison experiments is to employ pairs of options which have different levels for only some of the factors in a study. For the practically important case where the factors fall into three groups such that all factors within a group have the same number of levels and where one is only interested in estimating the main effects, a comprehensive catalogue of D-optimal approximate designs is presented. These optimal designs use at most three different types of pairs and have a block diagonal information matrix.

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1. Introduction

In a typical paired comparison experiment for eliciting preferences [see 1, 2], respondents are asked to repeatedly choose their preferred option from pairs of competing alternatives. Choice experiments generalize this approach to larger choice sets of three or more alternatives. When the alternatives are presented in pairs, often also a different response format is used where the degree of preference for the preferred option has to be expressed on a numerical rating scale.

Whereas historically the options in these experiments were unstructured stimuli, such as different loudness levels in psychophysics, over time interest has shifted to experiments involving options which are characterized by a combination of levels of several attributes or factors. Many of these developments were driven by methodological advances in marketing and in particular the widespread use of conjoint analysis [3, 4] and discrete choice experiments [5] for measuring consumer preferences. More recently, those methods have also been widely applied in health economics [6], transportation and environmental economics.

The number of options in a paired comparison or choice experiment increases with the number of attributes and levels that are used for describing their characteristics. Grouping those options into pairs or larger choice sets results in a large number of potential sets a respondent may be asked to evaluate. Thus in order to design questionnaires with a practical number of questions the usefulness of experimental design ideas was recognized early on [7]. The influential papers [8, 9] spawned additional interest in the optimal design of paired comparison and choice

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experiments, both among marketing scientists and statisticians. Most of the ensuing developments at the turn of the century have been summarized in [10] and [11]. More recent contributions to the area can be found in [12–14], among others.

This paper considers a design problem which is motivated by considerations of task complexity [15, 16]. This notion refers to possibly detrimental behavioral effects on the response quality caused by the large amount of information that has to be processed when choice sets are large or when the competing options are characterized by many attributes. In particular, respondents may employ simplifying strategies in their preference judgments, such as focusing only on some characteristics.

In [17] it was investigated how the efficiency of a design depends on the choice set size. Broadly speaking, the results show that choice sets of size three or four often give considerably higher efficiency values than sets of size two, and that the efficiency gains that can be achieved by using even larger choice sets are usually negligible. These results are however purely technical in the sense that they do not take into account the additional cognitive processing requirements implied by larger choice sets. It therefore seems that the optimal size of choice sets is a question which should primarily be answered empirically. In practical applications, mostly pairs of options or choice sets of size three appear to be used.

In situations with many attributes, one approach to dealing with simplifying response strategies is to include these explicitly in the model of the response process [18]. Some promising design work for this type of model has been recently presented in [19]. Another approach is to avoid information overload and hence the use of simplifying strategies by using partial profiles [4, 20], that is incomplete descriptions of the options. Here within each choice set only a subset of the attributes in a study is used to characterize the options and information about the complete set of attributes is obtained by varying the subsets between choice sets. For some applications see [21–23], among others.

In what follows, we consider the problem of finding optimal designs for experiments involving pairs of partial profiles, where the utility or value of every single option is assumed to depend linearly on the main effects of a number of attributes or factors with a finite number of levels. Optimal designs for the case where all factors have the same number of levels were presented in [13], but the problem becomes more challenging for factors with different numbers of levels.

Within the framework of approximate designs and by using invariance considerations, the general form of the information matrix of D-optimal designs for this situation and a sufficient condition for design optimality were derived in [24]. Although these results are fully general, for arbitrary numbers of factors and levels no general method for generating optimal designs has been proposed so far. As shown in [25] some progress can be made by considering more restricted settings. More precisely, often the factors characterizing the options can be arranged into two or three groups such that the factors within each group have the same number of levels. For example, the application in [26] investigated consumer preferences for characteristics of the pork production process. This work involved twenty-four attributes, which in addition to taste and price covered aspects of pig feeding, breeding, farming, processing and retail. All attributes had two, three or four levels. Part of this study involved paired comparisons of partial profiles where the alternatives were characterized by seven attributes and preference judgements were collected on a rating scale.

For two groups of factors approximate and exact optimal designs with practical numbers of pairs have been presented in [25]. The results in the current paper cover all situations where the factors fall into three groups. We restrict the discussion to approximate designs and present a comprehensive catalogue of D-optimal designs, which can be used to assess the efficiency of any other approximate or exact design. The support of every optimal design in the catalogue contains at most three different types of pairs of partial profiles. As will be outlined at the end of the paper, knowledge of these types of pairs and their corresponding design weights can guide the construction of exact optimal designs and also inform optimal design algorithms.

2. Design region for pairs of partial profiles

We consider paired comparison experiments with $K \geq 2$ factors where for $k = 1, \ldots, K$ the k-th factor has v_k levels in $\mathcal{X}_k = \{1, \ldots, v_k\}$. The competing options $\mathbf{s} = (s_1, \ldots, s_K)$ and $\mathbf{t} = (t_1, \ldots, t_K)$ in every pair $(\mathbf{s}, \mathbf{t})$ are elements of the Cartesian product $\mathcal{X}_1 \times \ldots \times \mathcal{X}_K$. Typically, one of the options in a pair has to be considered or tasted first and therefore the pair $(\mathbf{s}, \mathbf{t})$, where without loss of generality $\mathbf{s}$ is the first and $\mathbf{t}$ is the second option, is distinguished from the pair $(\mathbf{t}, \mathbf{s})$ with the same options in the opposite order. The set $\mathcal{X} = \{(\mathbf{s}, \mathbf{t}) : \mathbf{s}, \mathbf{t} \in \mathcal{X}_1 \times \ldots \times \mathcal{X}_K\}$ contains all ordered pairs of options.

As already explained, pairs of partial profiles involve incomplete descriptions of the options. These descriptions use only $S < K$ of the factors, where S is chosen by the experimenter. The options in every pair $(\mathbf{s}, \mathbf{t})$ then have different levels for S of the K factors and it is assumed that $\mathbf{s}$ and $\mathbf{t}$ do not differ with respect to the $K - S$ remaining factors. The S factors for which the options have different levels do however vary between pairs. To formalize this idea we follow the approach in [24] and consider the set

$$\mathcal{X}^*(S) = \{(\mathbf{s}, \mathbf{t}) \in \mathcal{X} : s_k \neq t_k \text{ for exactly } S \text{ indices } k\}$$

as the design region.

We refer to the options in a pair $(\mathbf{s}, \mathbf{t}) \in \mathcal{X}^*(S)$ as partial profiles and call S the profile strength. Note that although the partial profiles in a pair $(\mathbf{s}, \mathbf{t}) \in \mathcal{X}^*(S)$ are combinations of K factor levels, when asking respondents for their preferences typically only the S factors for which $\mathbf{s}$ and $\mathbf{t}$ have different levels are presented. Alternatively, the set of pairs which differ in at most S factors could be considered. However, since for the design criterion considered here the same designs remain optimal in this more general setting [24, p. 115], interest can be restricted to $\mathcal{X}^*(S)$.

For presenting some of the later results it is useful to partition the design region $\mathcal{X}^*(S)$ into $M = C(K, S)$ mutually disjoint sets $\mathcal{G}_1, \ldots, \mathcal{G}_M$, where $C(K, S) = \frac{K!}{S!(K-S)!}$ represents the binomial coefficient. For $m = 1, \ldots, M$ these sets are defined by $\mathcal{G}_m = \{(\mathbf{s}, \mathbf{t}) : s_k \neq t_k \text{ for } k \in \mathcal{F}_m \text{ and } s_k = t_k \text{ for } k \notin \mathcal{F}_m\}$, where $\mathcal{F}_m$ denotes the m-th subset of size S of $\{1, \ldots, K\}$ in a suitable (e.g. lexicographical) order. Thus each set $\mathcal{G}_m$ corresponds to a different set of S factors for which the partial profiles have different levels. The size of $\mathcal{G}_m$ is equal to $|\mathcal{G}_m| = \prod_{k \notin \mathcal{F}_m} v_k \times \prod_{k \in \mathcal{F}_m} v_k(v_k - 1)$.

3. Paired comparison models

The different response formats that may be used in a paired comparison experiment give rise to different statistical models. If for each pair only the preferred option is recorded then usually the multinomial logit model [see, e.g., 5, p. 49] which can be recognized as a variant of the Bradley–Terry model [27] is used. For experiments such as the one reported in [26], where the responses are given on a rating scale, linear models are appropriate.

In general, the two response formats and corresponding models lead to different design problems. If, however, as in [28] it is assumed that the two options in each pair are chosen with the same probability, then the problem of finding a design which is optimal with respect to a criterion that is based on the Fisher information matrix in the multinomial logit model is equivalent to the corresponding design problem in a linear paired comparison model [see, e.g., 10, p. 99] with model equation

$$Y(\mathbf{s},\mathbf{t}) = (\mathbf{f}(\mathbf{s}) - \mathbf{f}(\mathbf{t}))^\top \boldsymbol{\beta} + \varepsilon(\mathbf{s},\mathbf{t}), \tag{1}$$

where $Y(\mathbf{s},\mathbf{t})$ represents the response to the pair $(\mathbf{s},\mathbf{t}) \in \mathcal{X}$. Positive values of $Y(\mathbf{s},\mathbf{t})$ indicate a preference for $\mathbf{s}$ and negative values a preference for $\mathbf{t}$. The components of the vector $\mathbf{f}$ are known regression functions and $\boldsymbol{\beta}$ is the parameter vector of interest. The error terms $\varepsilon(\mathbf{s},\mathbf{t})$ have mean zero and are assumed to be homoscedastic and uncorrelated for different responses.

As in [24, 25] we consider such a linear model for the situation where only the main effects of the levels of the K factors in the previous section are of interest. It is then convenient to partition $\mathbf{f} = (\mathbf{f}_1^\top, \ldots, \mathbf{f}_K^\top)^\top$ and $\boldsymbol{\beta} = (\boldsymbol{\beta}_1^\top, \ldots, \boldsymbol{\beta}_K^\top)^\top$ in accordance with the factorial structure of the experiment so that $\mathbf{f}(j_1, \ldots, j_K) = (\mathbf{f}_1(j_1)^\top, \ldots, \mathbf{f}_K(j_K)^\top)^\top$ for every factor level combination $(j_1, \ldots, j_K) \in \mathcal{X}_1 \times \ldots \times \mathcal{X}_K$. For every factor k the function $\mathbf{f}_k$ is defined on $\mathcal{X}_k$ and maps the levels of the factor into their effects-coded equivalents. More precisely, for every k and $j \in \mathcal{X}_k$ the vector $\mathbf{f}_k(j)$ is a column vector of length $p_k = v_k - 1$. If $j < v_k$, then $\mathbf{f}_k(j)$ is defined as the j-th unit vector of length p_k and if $j = v_k$ it is equal to $-\mathbf{1}_{p_k}$, where $\mathbf{1}_{p_k}$ is a column vector of ones. The component $\boldsymbol{\beta}_k$ of $\boldsymbol{\beta}$ for factor k contains p_k parameters which can be interpreted as in a K-way main effects ANOVA model [see 13] and which reflect the utility or importance of the levels of factor k in making preference judgements.

4. Optimal approximate designs

In what follows, we consider the main effects paired comparison model (1) and the design region $\mathcal{X}^*(S)$. A continuous or approximate design [see, e.g., 29] is by definition any probability measure ξ on $\mathcal{X}^*(S)$, whereas an exact design of size N is just a collection of N not necessarily distinct pairs in $\mathcal{X}^*(S)$. In this paper, we only consider approximate designs.

Most optimality criteria for comparing different designs are based on the information matrix, which for any approximate design ξ for model (1) is given by $\mathbf{M}(\xi) = \int (\mathbf{f}(\mathbf{s}) - \mathbf{f}(\mathbf{t}))(\mathbf{f}(\mathbf{s}) - \mathbf{f}(\mathbf{t}))^\top \xi(d(\mathbf{s},\mathbf{t}))$. Here we employ the D-optimality criterion which aims at maximizing the determinant $|\mathbf{M}(\xi)|$ of the information matrix. An approximate design ξ^* is then D-optimal, if $|\mathbf{M}(\xi^*)| \geq |\mathbf{M}(\xi)|$ for every other approximate design ξ.

The work in [24] for the partial profile design problem in the main effects model (1) considered approximate designs of the form

$$\xi = \sum_{m=1}^{M} \sum_{(\mathbf{s},\mathbf{t}) \in \mathcal{G}_m} \frac{w_m}{|\mathcal{G}_m|} \xi_{(\mathbf{s},\mathbf{t})} \tag{2}$$

where $w_1, \ldots, w_M$ are nonnegative weights summing to one, $\xi_{(\mathbf{s},\mathbf{t})}$ denotes the onepoint probability measure concentrated in $(\mathbf{s}, \mathbf{t})$, and all other terms have been defined in Section 2. Every such design ξ has a block diagonal information matrix

$$\mathbf{M}(\xi) = \begin{pmatrix} c_1(\xi)\mathbf{M}_{v_1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & c_K(\xi)\mathbf{M}_{v_K} \end{pmatrix}, \tag{3}$$

where $c_k(\xi) = \sum_{1 \le m \le M : k \in \mathcal{F}_m} w_m$ for every k and $\mathbf{M}_a = \frac{2}{a-1}(\mathbf{I}_{a-1} + \mathbf{1}_{a-1}\mathbf{1}_{a-1}^\top)$ for every positive integer $a > 1$. Note that $c_k(\xi)$ is the sum of the weights w_m associated with those subsets $\mathcal{G}_m$ of the design region for which the options in a pair have different levels for factor k.

The results in [24] show that there always exists a D-optimal design of the form (2). Moreover, the paper proved a sufficient condition which states that every design ξ in (2) for which $c_k(\xi) = p_k S/p$, $k = 1, \ldots, K$, is D-optimal, where $p = \sum_{k=1}^K p_k$ is the total number of parameters in model (1). This condition is however not necessary and there are situations where the constants $c_k(\xi)$ corresponding to a D-optimal design ξ of the form (2) are not all equal to $p_k S/p$.

For general numbers of factors K, numbers of levels v_k, $k = 1, \ldots, K$, and profile strength S, determining an optimal design of the form in (2) is not a trivial problem. Firstly, although in most cases the sufficient condition in [24] for the constants $c_k(\xi)$ in (3) can be applied to verify the D-optimality of a suitably chosen design, it may be challenging to identify and deal with those cases which are not covered by this result. Secondly, even when a D-optimal design ξ of the form (2) with $c_k(\xi) = p_k S/p$ for $k = 1, \ldots, K$ exists, it is usually not easy to identify the support points for which the corresponding weights in (2) are positive and to calculate those weights. As has already been indicated, progress can be made by restricting the discussion to situations where there are only a few groups of factors with different numbers of levels. The next section generalizes the approach of [25] for two groups to three groups of factors.

5. Optimal designs for three groups of factors

Suppose that the K factors in Section 2 fall into three groups such that for $j = 1, 2, 3$ the $K_j > 0$ factors in group j have the same number of u_j levels, where without loss of generality it can be assumed that $u_1 < u_2 < u_3$. More specifically, it can be assumed that the $K = K_1 + K_2 + K_3$ factors are ordered in such a way that for factor k with $\sum_{a=1}^{j-1} K_a + 1 \le k \le \sum_{a=1}^{j} K_a$ the number of levels is equal to $v_k = u_j$, $j = 1, 2, 3$. The number of model parameters corresponding to attributes in group j is $q_j = u_j - 1$. In other words, $p_k = q_j$ for all k with $\sum_{a=1}^{j-1} K_a + 1 \le k \le \sum_{a=1}^{j} K_a$, $j = 1, 2, 3$. The total number of model parameters then equals $p = \sum_{j=1}^3 K_j q_j$.

For a given profile strength S, a set $\mathcal{G}_m$ as defined in Section 2 is said to be of type (n_1, n_2, n_3) where $n_1 + n_2 + n_3 = S$, if the corresponding subset $\mathcal{F}_m \subset \{1, \ldots, K\}$ in Section 2 contains $n_j \le K_j$ factors with u_j levels where $j = 1, 2, 3$. Thus if $(\mathbf{s}, \mathbf{t})$ belongs to a set $\mathcal{G}_m$ of type (n_1, n_2, n_3) then $\mathbf{s}$ and $\mathbf{t}$ have different levels for n_1 factors with u_1 levels, n_2 factors with u_2 levels and n_3 factors with u_3 levels. In total there are $\prod_{j=1}^3 C(K_j, n_j)$ sets $\mathcal{G}_m$ of type (n_1, n_2, n_3), where again $C(K_j, n_j)$ denotes the binomial coefficient. Moreover, for every set $\mathcal{G}_m$ of type (n_1, n_2, n_3) the expression for $|\mathcal{G}_m|$ in Section 2 simplifies to $|\mathcal{G}_m| = \prod_{j=1}^3 u_j^{K_j}(u_j - 1)^{n_j}$.

The number of sets $\mathcal{G}_m$ of a given type (n_1, n_2, n_3) for which the options in each

Table 1. Main cases

Case	$K_1 \geq S$	$K_2 \geq S$	$K_3 \geq S$	Case	$K_1 \geq S$	$K_2 \geq S$	$K_3 \geq S$
1)	+[a]	+	+	5)	−	−	+
2)	−	+	+	6)	−	+	−
3)	+	−	+	7)	+	−	−
4)	+	+	−	8)	−	−	−

[a] A '+' indicates that the condition $K_j \geq S$ in the heading of a column is true, whereas a '−' means that $K_j < S$.

pair $(\mathbf{s}, \mathbf{t}) \in \mathcal{G}_m$ have different levels for a particular factor from group j is the same regardless of which factor in the group is considered. It follows that if in (2) the same weight $w_m = w_{(n_1,n_2,n_3)}$ is assigned to all sets $\mathcal{G}_m$ which are of the same type (n_1, n_2, n_3), then $c_k(\xi)$ in (3) is the same for every factor k in group j and we denote this common value by $c^{(j)}(\xi)$. Such designs are appealing since they are invariant with respect to permutations of factors which have the same number of levels. The corresponding information $\mathbf{M}(\xi)$ in (3) then has K_j, $j = 1, 2, 3$, diagonal blocks equal to $c^{(j)}(\xi)\mathbf{M}_{u_j}$.

In what follows we present D-optimal designs ξ of the form (2) where $w_m = w_{(n_1,n_2,n_3)}$ is the same for every set $\mathcal{G}_m$ of type (n_1, n_2, n_3). In order to facilitate the presentation we denote the union of all sets $\mathcal{G}_m$ of type (n_1, n_2, n_3) by $\mathcal{G}^{(n_1,n_2,n_3)}$ and the corresponding weight $\xi(\mathcal{G}^{(n_1,n_2,n_3)})$ by $w^{(n_1,n_2,n_3)}$. It follows that

$$w^{(n_1,n_2,n_3)} = w_{(n_1,n_2,n_3)} \prod_{j=1}^{3} C(K_j, n_j). \tag{4}$$

The results are depicted in Table 4 in the form of a catalogue of designs. The optimal designs strongly depend on the numbers of factors and levels in each of the three groups as well as on the profile strength. This is reflected by the fact that in Table 4 a total of 49 cases have to be considered in order to be able to deal with all situations. Although we tried to get by with as few cases as possible, no claim can be made that this goal has been achieved. The possible situations may be enumerated in a different way, though we conjecture that still a large number of cases would have to be distinguished. At least, however, it can be easily checked that the catalogue is complete in the sense that it provides coverage of all situations.

The presentation in Table 4 is organized according to how the group sizes K_1, K_2, K_3 relate to the profile strength $S < K$. In total, the eight main cases shown in Table 1 need to be considered. However, this partition is too coarse and further subcases have to be distinguished. For example, in case 8) where the size of every group is smaller than the profile strength the finer distinctions shown in Table 2 are necessary. Often, still additional conditions have to be employed which further increase the number of cases. These additional conditions are given in Table 4.

Table 2. Subcases when $K_1, K_2, K_3 < S$

Case	$K_1 + K_2 < S$	$K_1 + K_3 < S$	$K_2 + K_3 < S$	Case	$K_1 + K_2 < S$	$K_1 + K_3 < S$	$K_2 + K_3 < S$
8a)	+[a]	+	+	8e)	−	+	+
8b)	+	+	−	8f)	−	+	−
8c)	+	−	+	8g)	−	−	+
8d)	+	−	−	8h)	−	−	−

[a] A '+' indicates that the condition $K_i + K_j < S$ in the heading of a column is true, whereas a '−' means that $K_i + K_j \geq S$.

Table 3. Constants $c^{(j)}(\xi)$ for D-optimal designs

Number of design	$c^{(1)}(\xi)$	$c^{(2)}(\xi)$	$c^{(3)}(\xi)$
6, 10, 12, 16, 17, 23, 26, 30, 33, 37, 40, 44, 49	$\frac{q_1(S-K_3)}{K_1q_1+K_2q_2}$	$\frac{q_2(S-K_3)}{K_1q_1+K_2q_2}$	1
15, 22, 29, 36, 43	$1-\frac{K-S}{K_1}$	1	1

For every design the catalogue depicts the types (n_1, n_2, n_3) of the $\mathcal{G}_m$ in its support and the corresponding weights $w^{(n_1,n_2,n_3)}$ in (4), which in every case add up to one. Whenever the expression for a weight $w^{(n_1,n_2,n_3)}$ involves a difference of the form $K_i + K_j - S$ in the denominator of a fraction, the additional conditions in Table 4 ensure that the difference is positive and hence that the weight is well-defined. The weights $w_{(n_1,n_2,n_3)}$ for individual sets $\mathcal{G}_m$ of the specified type (n_1, n_2, n_3) can then be easily obtained from (4). For all other $\mathcal{G}_m$ not listed in the catalogue the weight in (2) is $w_m = 0$. In general, all weights presented are positive. Yet, for some constellations the expression given for one or the other weight can be equal to zero. But even in those cases the information matrix of the corresponding design is still regular.

Most of the designs in Table 4 can be shown to be D-optimal by using the sufficient condition in [24], that is by verifying that $c^{(j)}(\xi) = q_jS/p$ for $j = 1, 2, 3$. When this result cannot be applied, the variance function has to be examined more carefully and optimality follows from the Kiefer–Wolfowitz Equivalence Theorem [30]. More precisely, in such cases it has to be shown that the variance function

$$d((\mathbf{s},\mathbf{t});\xi) = (\mathbf{f}(\mathbf{s}) - \mathbf{f}(\mathbf{t}))^\top \mathbf{M}(\xi)^{-1}(\mathbf{f}(\mathbf{s}) - \mathbf{f}(\mathbf{t}))$$

of a design ξ in the catalogue is bounded from above on the design region $\mathcal{X}^*(S)$ by the number p of model parameters, where $\mathbf{f}$ is the vector of regression functions in model (1). If a pair $(\mathbf{s},\mathbf{t})$ belongs to a set $\mathcal{G}_m$ of type (n_1, n_2, n_3) then by using an expression for $d((\mathbf{s},\mathbf{t});\xi)$ from [24] it is not difficult to see that

$$d((\mathbf{s},\mathbf{t});\xi) = \frac{n_1q_1}{c^{(1)}(\xi)} + \frac{n_2q_2}{c^{(2)}(\xi)} + \frac{n_3q_3}{c^{(3)}(\xi)}.$$

When $c^{(j)}(\xi) = q_jS/p$ for $j = 1, 2, 3$, then the above equation immediately shows that $d((\mathbf{s},\mathbf{t});\xi) = p$ for all $(\mathbf{s},\mathbf{t}) \in \mathcal{X}^*(S)$, which proves the D-optimality of the design. Table 3 reports the constants $c^{(j)}(\xi)$ derived from Table 4 which are needed for verifying that $d((\mathbf{s},\mathbf{t});\xi) \le p$ in all other cases. Due to space limitations we omit the tedious details.

6. Discussion

Each of the optimal designs in Table 4 uses at most three different types of pairs and the corresponding weights $w^{(n_1,n_2,n_3)}$ indicate the proportions of each type in the design. The information matrix of every design ξ in the catalogue is of the form (3). The diagonal blocks for the first K_1 factors with u_1 levels are equal to $c^{(1)}(\xi)\mathbf{M}_{u_1}$, and for the next K_2 factors with u_2 levels and the remaining K_3 factors with u_3 levels they are given by $c^{(2)}(\xi)\mathbf{M}_{u_2}$ and $c^{(3)}(\xi)\mathbf{M}_{u_3}$, respectively. For most designs in Table 4 it holds that $c^{(j)}(\xi) = q_jS/p$, $j = 1, 2, 3$ and for all remaining cases the constants are given in Table 3.

Thus the information matrix and corresponding determinant of any design in the catalogue can be easily calculated without having to pay much attention to the

Table 4. Catalogue of optimal designs

Case	Additional conditions	Design	Pair types (n_1, n_2, n_3)	Weights $w^{(n_1,n_2,n_3)}$
1)		1	$(S,0,0)$	$\frac{K_1q_1}{p}$
			$(0,S,0)$	$\frac{K_2q_2}{p}$
			$(0,0,S)$	$\frac{K_3q_3}{p}$
2)		2	$(K_1,S-K_1,0)$	$\frac{Sq_1}{p}$
			$(0,S,0)$	$\frac{K_2q_2-(S-K_1)q_1}{p}$
			$(0,0,S)$	$\frac{K_3q_3}{p}$
3)		3	$(S,0,0)$	$\frac{K_1q_1}{p}$
			$(0,K_2,S-K_2)$	$\frac{Sq_2}{p}$
			$(0,0,S)$	$\frac{K_3q_3-(S-K_2)q_2}{p}$
4)	$Sq_3 \le K_2q_2 + K_3q_3$	4	$(S,0,0)$	$\frac{K_1q_1}{p}$
			$(0,S,0)$	$\frac{K_2q_2-(S-K_3)q_3}{p}$
			$(0,S-K_3,K_3)$	$\frac{Sq_3}{p}$
	$K_2q_2 + K_3q_3 < Sq_3 \le p$	5	$(S,0,0)$	$1-\frac{Sq_3}{p}$
			$(S-K_3,0,K_3)$	$\frac{Sq_3}{p}-\frac{SK_2q_2}{(S-K_3)p}$
			$(0,S-K_3,K_3)$	$\frac{SK_2q_2}{(S-K_3)p}$
	$Sq_3 > p$	6	$(S-K_3,0,K_3)$	$\frac{K_1q_1}{K_1q_1+K_2q_2}$
			$(0,S-K_3,K_3)$	$\frac{K_2q_2}{K_1q_1+K_2q_2}$
5)	$S(q_1+q_2) \le p$	7	$(K_1,0,S-K_1)$	$\frac{Sq_1}{p}$
			$(0,K_2,S-K_2)$	$\frac{Sq_2}{p}$
			$(0,0,S)$	$1-\frac{S(q_1+q_2)}{p}$
	$S(q_1+q_2) > p$,	8	$(0,K_2,S-K_2)$	$\frac{S(q_2-q_1)}{p}$
	$K_1+K_2 < S$		$(K_1,K_2,S-K_1-K_2)$	$\frac{Sq_1}{p}$
			$(0,0,S)$	$1-\frac{Sq_2}{p}$
	For 5) the case $S(q_1+q_2) > p$, $K_1+K_2 \ge S$ is not possible.			
6)	$S(q_1+q_3) \le p$	9	$(K_1,S-K_1,0)$	$\frac{Sq_1}{p}$
			$(0,S,0)$	$1-\frac{S(q_1+q_3)}{p}$
			$(0,S-K_3,K_3)$	$\frac{Sq_3}{p}$
	$Sq_3 > p$,	10	$(0,S-K_3,K_3)$	$1-\frac{(S-K_3)q_1}{K_1q_1+K_2q_2}$
	$K_1+K_3 \le S$		$(K_1,S-K_1-K_3,K_3)$	$\frac{(S-K_3)q_1}{K_1q_1+K_2q_2}$
	$p-Sq_1 < Sq_3 \le p$,	11	$(K_1,S-K_1,0)$	$1-\frac{Sq_3}{p}$
	$K_1+K_3 \le S$		$(K_1,S-K_1-K_3,K_3)$	$\frac{S(q_1+q_3)}{p}-1$
			$(0,S-K_3,K_3)$	$1-\frac{Sq_1}{p}$
	$Sq_3 > p$, $K_1+K_3 > S$	12	Design no. 6 is optimal.	
	$p-Sq_1 < Sq_3 \le p$,	13	$(S-K_3,0,K_3)$	$\frac{SK_1q_1}{(S-K_3)p}$
	$K_1+K_3 > S$		$(0,S,0)$	$1-\frac{Sq_3}{p}$
			$(0,S-K_3,K_3)$	$\frac{Sq_3}{p}-\frac{SK_1q_1}{(S-K_3)p}$
7)	$S(q_2+q_3) \le p$	14	$(S,0,0)$	$1-\frac{S(q_2+q_3)}{p}$
			$(S-K_2,K_2,0)$	$\frac{Sq_2}{p}$
			$(S-K_3,0,K_3)$	$\frac{Sq_3}{p}$
	$Sq_3 > p$, $K_2+K_3 < S$, $\frac{q_2(S-K_2-K_3)}{K_1q_1} \ge 1$	15	$(S-K_2-K_3,K_2,K_3)$	1
	$Sq_3 > p$, $K_2+K_3 < S$,	16	$(S-K_3,0,K_3)$	$1-\frac{(S-K_3)q_2}{K_1q_1+K_2q_2}$
	$\frac{q_2(S-K_2-K_3)}{K_1q_1} < 1$		$(S-K_2-K_3,K_2,K_3)$	$\frac{(S-K_3)q_2}{K_1q_1+K_2q_2}$
	$Sq_3 > p$, $K_2+K_3 \ge S$	17	Design no. 6 is optimal.	
	$p-Sq_2 < Sq_3 \le p$,	18	$(S-K_2,K_2,0)$	$1-\frac{Sq_3}{p}$
	$K_2+K_3 \le S$		$(S-K_3,0,K_3)$	$1-\frac{Sq_2}{p}$
			$(S-K_2-K_3,K_2,K_3)$	$\frac{S(q_2+q_3)}{p}-1$
	$p-Sq_2 < Sq_3 \le p$,	19	$(S-K_2,K_2,0)$	$1-\frac{Sq_3}{p}$
	$K_2+K_3 > S$,		$(S-K_3,0,K_3)$	$\frac{Sq_3}{p}-\frac{K_2}{S-K_3}\left(\frac{(q_2+q_3)S}{p}-1\right)$
	$\frac{(S-K_3)Sq_3}{K_2(S(q_2+q_3)-p)} > 1$		$(0,S-K_3,K_3)$	$\frac{K_2}{S-K_3}\left(\frac{(q_2+q_3)S}{p}-1\right)$
	$p-Sq_2 < Sq_3 \le p$,	20	$(S-K_2,K_2,0)$	$\frac{SK_1q_1}{(S-K_2)p}$
	$K_2+K_3 > S$,		$(0,K_2,S-K_2)$	$\frac{K_3}{S-K_2}\frac{K_2(S(q_2+q_3)-p)-(S-K_3)Sq_3}{(K_2+K_3-S)p}$
	$\frac{(S-K_3)Sq_3}{K_2(S(q_2+q_3)-p)} \le 1$		$(0,S-K_3,K_3)$	$\frac{K_2}{K_2+K_3-S}\left(1-\frac{Sq_2}{p}\right)$

Table 4. Catalogue of optimal designs (continued)

Case	Additional conditions	Design	Pair types (n_1, n_2, n_3)	Weights $w^{(n_1,n_2,n_3)}$
8a)	$Sq_3 \le p$	21	$(K_1, K_2, S-K_1-K_2)$	$\frac{K_3}{K-S}\left(1-\frac{Sq_3}{p}\right)$
			$(K_1, S-K_1-K_3, K_3)$	$\frac{K_2}{K-S}\left(1-\frac{Sq_2}{p}\right)$
			$(S-K_2-K_3, K_2, K_3)$	$\frac{K_1}{K-S}\left(1-\frac{Sq_1}{p}\right)$
	$Sq_3 > p$, $\frac{(S-K_2-K_3)q_2}{K_1q_1} \ge 1$	22	Design no. 15 is optimal.	
	$Sq_3 > p$, $\frac{(S-K_2-K_3)q_2}{K_1q_1} < 1$	23	$(K_1, S-K_1-K_3, K_3)$	$\frac{K_2}{K-S}\left(1-\frac{(S-K_3)q_2}{K_1q_1+K_2q_2}\right)$
			$(S-K_2-K_3, K_2, K_3)$	$\frac{K_1}{K-S}\left(1-\frac{(S-K_3)q_1}{K_1q_1+K_2q_2}\right)$
8b)	$Sq_3 \le p$, $K_3(p-Sq_3) \ge (K-S)Sq_1$	24	$(K_1, K_2, S-K_1-K_2)$	$\frac{Sq_1}{p}$
			$(0, K_2, S-K_2)$	$1-\frac{Sq_1}{p}-\frac{K_2}{K_2+K_3-S}\left(1-\frac{Sq_2}{p}\right)$
			$(0, S-K_3, K_3)$	$\frac{K_2}{K_2+K_3-S}\left(1-\frac{Sq_2}{p}\right)$
	$Sq_3 \le p$, $K_3(p-Sq_3) < (K-S)Sq_1$	25	$(K_1, K_2, S-K_1-K_2)$	$\frac{K_3}{K-S}\left(1-\frac{Sq_3}{p}\right)$
			$(K_1, S-K_1-K_3, K_3)$	$\frac{Sq_1}{p}-\frac{K_3}{K-S}\left(1-\frac{Sq_3}{p}\right)$
			$(0, S-K_3, K_3)$	$1-\frac{Sq_1}{p}$
	$Sq_3 > p$	26	Design no. 10 is optimal.	
8c)	$Sq_3 \le p$, $K_1(p-Sq_1) > (K-S)Sq_2$	27	$(K_1, 0, S-K_1)$	$\frac{K_3}{K_1+K_3-S}\left(1-\frac{Sq_3}{p}\right)$
			$(S-K_3, 0, K_3)$	$\frac{K_1}{K_1+K_3-S}\left(1-\frac{Sq_1}{p}-\frac{K-S}{K_1}\frac{Sq_2}{p}\right)$
			$(S-K_2-K_3, K_2, K_3)$	$\frac{Sq_2}{p}$
	$Sq_3 \le p$, $K_1(p-Sq_1) \le (K-S)Sq_2$	28	$(K_1, 0, S-K_1)$	$1-\frac{Sq_2}{p}$
			$(K_1, K_2, S-K_1-K_2)$	$\frac{Sq_2}{p}-\frac{K_1}{K-S}\left(1-\frac{Sq_1}{p}\right)$
			$(S-K_2-K_3, K_2, K_3)$	$\frac{K_1}{K-S}\left(1-\frac{Sq_1}{p}\right)$
	$Sq_3 > p$, $\frac{(S-K_2-K_3)q_2}{K_1q_1} \ge 1$	29	Design no. 15 is optimal.	
	$Sq_3 > p$, $\frac{(S-K_2-K_3)q_2}{K_1q_1} < 1$	30	Design no. 16 is optimal.	
8d)	$Sq_3 \le p$, $K_3(p-Sq_3) \ge (K-S)Sq_1$	31	Design no. 24 is optimal.	
	$Sq_3 \le p$, $K_3(p-Sq_3) < (K-S)Sq_1$	32	$(K_1, K_2, S-K_1-K_2)$	$\frac{K_3}{K-S}\left(1-\frac{Sq_3}{p}\right)$
			$(S-K_3, 0, K_3)$	$\frac{K_1}{S-K_3}\left(\frac{Sq_1}{p}-\frac{K_3}{K-S}\left(1-\frac{Sq_3}{p}\right)\right)$
			$(0, S-K_3, K_3)$	$\frac{K_2}{K-S}\left(\frac{K_1}{S-K_3}\frac{S(q_2-q_1)}{p}+1-\frac{Sq_2}{p}\right)$
	$Sq_3 > p$	33	Design no. 6 is optimal.	
8e)	$Sq_3 \le p$, $K_2(p-Sq_2) \ge (K-S)Sq_3$	34	$(K_1, S-K_1, 0)$	$1-\frac{Sq_3}{p}-\frac{K_1}{K_1+K_2-S}\left(1-\frac{Sq_1}{p}\right)$
			$(S-K_2, K_2, 0)$	$\frac{K_1}{K_1+K_2-S}\left(1-\frac{Sq_1}{p}\right)$
			$(K_1, S-K_1-K_3, K_3)$	$\frac{Sq_3}{p}$
	$Sq_3 \le p$, $K_2(p-Sq_2) < (K-S)Sq_3$	35	$(S-K_2, K_2, 0)$	$1-\frac{Sq_3}{p}$
			$(K_1, S-K_1-K_3, K_3)$	$\frac{K_2}{K-S}\left(1-\frac{Sq_2}{p}\right)$
			$(S-K_2-K_3, K_2, K_3)$	$\frac{Sq_3}{p}-\frac{K_2}{K-S}\left(1-\frac{Sq_2}{p}\right)$
	$Sq_3 > p$, $\frac{(S-K_2-K_3)q_2}{K_1q_1} \ge 1$	36	Design no. 15 is optimal.	
	$Sq_3 > p$, $\frac{(S-K_2-K_3)q_2}{K_1q_1} < 1$	37	Design no. 23 is optimal.	
8f)	$Sq_3 \le p$, $K_2(p-Sq_2) > (K-S)Sq_1$	38	$(K_1, S-K_1-K_3, K_3)$	$\frac{Sq_1}{p}$
			$(0, K_2, S-K_2)$	$\frac{K_3}{K_2+K_3-S}\left(1-\frac{Sq_3}{p}\right)$
			$(0, S-K_3, K_3)$	$1-\frac{Sq_1}{p}-\frac{K_3}{K_2+K_3-S}\left(1-\frac{Sq_3}{p}\right)$
	$Sq_3 \le p$, $K_2(p-Sq_2) \le (K-S)Sq_1$	39	$(K_1, S-K_1-K_3, K_3)$	$\frac{K_2}{K-S}\left(1-\frac{Sq_2}{p}\right)$
			$(S-K_2, K_2, 0)$	$\frac{K_1}{S-K_2}\left(\frac{Sq_1}{p}-\frac{K_2}{K-S}\left(1-\frac{Sq_2}{p}\right)\right)$
			$(0, K_2, S-K_2)$	$\frac{K_3}{K-S}\left(\frac{K_1}{S-K_2}\frac{S(q_3-q_1)}{p}+1-\frac{Sq_3}{p}\right)$
	$Sq_3 > p$	40	Design no. 10 is optimal.	

Table 4. Catalogue of optimal designs (continued)

Case	Additional conditions	Design	Pair types (n_1, n_2, n_3)	Weights $w^{(n_1,n_2,n_3)}$
8g)	$Sq_3 \leq p$, $\frac{K_1(p-Sq_1)}{(K-S)Sq_2} > 1$	41	Design no. 27 is optimal.	
	$Sq_3 \leq p$, $\frac{K_1(p-Sq_1)}{(K-S)Sq_2} \leq 1$	42	$(S-K_2-K_3, K_2, K_3)$	$\frac{K_1}{K-S}\left(1-\frac{Sq_1}{p}\right)$
			$(K_1, S-K_1, 0)$	$\frac{K_2}{S-K_1}\left(\frac{Sq_2}{p}-\frac{K_1}{K-S}\left(1-\frac{Sq_1}{p}\right)\right)$
			$(K_1, 0, S-K_1)$	$\frac{K_3}{S-K_1}\left(\frac{Sq_3}{p}-\frac{K_1}{K-S}\left(1-\frac{Sq_1}{p}\right)\right)$
	$Sq_3 > p$, $\frac{(S-K_2-K_3)q_2}{K_1q_1} \geq 1$	43	Design no. 15 is optimal.	
	$Sq_3 > p$, $\frac{(S-K_2-K_3)q_2}{K_1q_1} < 1$	44	Design no. 16 is optimal.	
8h)	$S(q_2+q_3) \leq p$, $\frac{K_1(p-Sq_1)}{(K_1+K_3-S)Sq_3} > 1$	45	$(K_1, S-K_1, 0)$	$\frac{K_2}{K_1+K_2-S}\left(1-\frac{S(q_2+q_3)}{p}\right)$
			$(S-K_2, K_2, 0)$	$\frac{K_1}{K_1+K_2-S}\left(1-\frac{Sq_1}{p}\right)-\frac{K_1+K_3-S}{K_1+K_2-S}\frac{Sq_3}{p}$
			$(S-K_3, 0, K_3)$	$\frac{Sq_3}{p}$
	$S(q_2+q_3) \leq p$, $\frac{K_1(p-Sq_1)}{(K_1+K_3-S)Sq_3} \leq 1$	46	$(K_1, S-K_1, 0)$	$\frac{K_2}{S-K_1}\frac{Sq_2}{p}$
			$(K_1, 0, S-K_1)$	$1-\frac{K_2}{S-K_1}\frac{Sq_2}{p}-\frac{K_1}{K_1+K_3-S}\left(1-\frac{Sq_1}{p}\right)$
			$(S-K_3, 0, K_3)$	$\frac{K_1}{K_1+K_3-S}\left(1-\frac{Sq_1}{p}\right)$
	$p-Sq_2 < Sq_3 \leq p$, $\frac{(S-K_3)Sq_3}{K_2(S(q_2+q_3)-p)} > 1$	47	Design no. 19 is optimal.	
	$p-Sq_2 < Sq_3 \leq p$, $\frac{(S-K_3)Sq_3}{K_2(S(q_2+q_3)-p)} \leq 1$	48	Design no. 20 is optimal.	
	$Sq_3 > p$	49	Design no. 6 is optimal.	

details in Table 4. Although, in general, D-optimal designs are not unique, their information matrices and hence determinants are, and so the corresponding values for the designs in the catalogue provide a benchmark for assessing any competing design. In particular, they may be used to assess the quality of any exact design ξ_N of size N in the usual way by means of comparing the determinant of the matrix $\frac{1}{N}\mathbf{X}^\top\mathbf{X}$, where $\mathbf{X}$ is the design matrix for ξ_N in model (1), with $|\mathbf{M}(\xi)|$ of the relevant design ξ in the catalogue.

Moreover, the designs in Table 4 are also D-optimal for the multinomial logit model with choice sets of size two, if as in [28] it is assumed that each option in a pair is chosen with the same probability. If the attribute levels in the choice model are effects-coded, then the information matrices of the designs in Table 4 can again be used as benchmarks for assessing alternative choice designs. It has however to be noted that in this case the information matrix in the multinomial logit model of every design ξ in the table is equal to $\frac{1}{4}\mathbf{M}(\xi)$, so that when using the optimal designs presented here as benchmarks the additional factor $1/4$ needs to be taken into account [10, 31].

In principle, the optimal designs in the catalogue can be implemented as exact designs, but the required numbers of pairs would be prohibitively large. Thus for practical purposes reduction methods for generating efficient exact designs are needed. Alternatively, standard exchange algorithms may be used. In both cases, the information in Table 4 about the types of pairs and the corresponding weights can be useful. For example, efficient exact designs may be constructed by using only pairs of the types specified in the catalogue and in relative proportions given by the weights in Table 4. For the case of two groups of factors corresponding results for exact designs have been presented in [25], but the same approach can also be used here. Likewise, algorithmic searches for optimal designs may benefit from employing starting designs which use the types of pairs and the weights in the catalogue.

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